# TORSION IN THE CHOW GROUP OF CODIMENSION TWO: THE CASE OF VARIETIES WITH ISOLATED SINGULARITIES 

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One of the most interesting applications of the recent results of Merkuriev and Suslin on $K_{2}$ is
0.1. Theorem [10], [7]. Let $X$ be a non-singular quasi-projective variety defined over an algebraically closed field, let $n$ be an integer prime to the characteristic, then ${ }_{n} \mathrm{CH}^{2}(X)$ is finite.

Here $\mathrm{CH}^{2}(X)$ denotes the group of codimension 2 cycles modulo rational equivalence and ${ }_{n} \mathrm{CH}^{2}(X)$ is the kernel of the multiplication by $n$. Our aim is to extend the theorem to the case of varieties with isolated singularities.

To begin with one has to decide what the second Chow group should be in this case. Let $Y$ be an irreducible quasi-projective variety with finitely nany singular points $y_{1}, y_{2}, \ldots, y_{n}$; we write $Y_{i}=$ set of points (i.e. irreducible cycles) of codimension $i$ in $Y$ and $Y_{i}^{+}=\left\{y \in Y_{i}:\left\{y_{1}, y_{2}, \ldots, y_{n}\right\} \cap \bar{y}=\emptyset\right\}$. Let

$$
C^{+i}=\coprod_{y \in Y_{i}^{*}} \mathbb{Z}_{y}
$$

We define $R^{+i}$ to be the subgroup of $C^{+i}$ generated by the elements of the form $(s, f)$, where $s \in Y_{i-1}^{+}, f$ is an element of $k(s)^{*}$, the group of invertible elements in the function field of $s$, and $(s, f)$ denotes the cycle $\left((f)_{0}-(f)_{\infty}\right)$ computed on $Y$. We define $\mathrm{CH}^{2}(Y)=C^{+2} / R^{+2}$. By [5, (6.2)] $\mathrm{CH}^{2}(Y) \simeq G^{2} K_{0} Y$, where $K_{0} Y$ is the Grothendieck group of vector bundles on $Y$ and $G^{2} K_{0} Y$ is the second graded group associated with a natural topological filtration on $K_{0} Y$. We shall prove

### 0.2. Theorem. ${ }_{n} \mathrm{CH}^{2}(Y)$ is finite.

Let $X$ be obtained from $Y$ by identifying all the singular points to one point $x_{0}$,

[^0]then $\mathrm{CH}^{2}(Y)=\mathrm{CH}^{2}(X)$, simply by definition, while $\mathrm{CH}^{2}(X)=H^{2}\left(X, \mathrm{~K}_{2}\right)$, [5]. To prove Theorem 0.2 we shall in fact establish the equivalent
0.3. Theorem. Let $X$ be a quasi-projective variety with one singular point at most, let I be a prime different from the characteristic of the base field, then $l^{10} H^{2}\left(X, K_{2}\right)$ is finite for every natural number $v$.

We fix an algebraically closed field as ground field for all varieties considered herein. We shall use the standard notations of [2] and [7], in particular $\mu_{l^{v}}^{\otimes n}$ shall denote the étale sheaf of $l^{v}$ roots of one tensored with itself $n$-times. For simplicity we write sometime $m$ or $m(v)$ instead of $l^{v}$.
1.

Let $X$ be a variety, $x_{0}$ a distinguished closed point on it such that $X-\left\{x_{0}\right\}$ is non-singular, there is an exact sequence [5]:
(GR*)

$$
0 \rightarrow \mathbf{K}_{2 X} \rightarrow \mathbf{K}_{2}\left(X_{x_{0}}\right) \xrightarrow{R} \coprod_{x \in X_{i}^{*}}\left(i_{x}\right)_{*} k(x)^{*} \xrightarrow{\partial} \coprod_{x \in X_{2}^{*}}\left(i_{x}\right)_{*} \mathbb{Z}_{x} \rightarrow 0 .
$$

Also the following Gersten complex is exact over $X-\left\{x_{0}\right\}$ by [11]:
(GR)

$$
0 \rightarrow \mathbf{K}_{2 X} \rightarrow\left(i_{X}\right)_{*} \mathbf{K}_{2} k(X) \xrightarrow{T} \coprod_{x \in X_{1}}\left(i_{x}\right)_{*} k(x)^{*} \xrightarrow{\partial} \coprod_{x \in X_{2}}\left(i_{x}\right)_{*} \mathbb{Z}_{x} \rightarrow 0
$$

l.et $\mathbf{F}$ be the subsheaf of $\mathrm{U}_{x \in X_{1}}\left(i_{x}\right)_{*} k(x)^{*}$ generated by $T\left(\left(i_{X}\right)_{*} \mathbf{K}_{2} k(X)\right)$ and by $\coprod_{x \in X ;}\left(i_{x}\right)_{*} k(x)^{*}$. We let $\mathbf{G}_{2 X}$ be the kernel of $T$ in the complex

$$
\begin{equation*}
0 \rightarrow \mathbf{G}_{2 X} \rightarrow\left(i_{X}\right)_{*} K_{2} k(X) \xrightarrow{T} \mathbf{F} \xrightarrow{\partial} \coprod_{x \in X_{i}}\left(i_{x}\right)_{*} \mathbb{Z}_{x} \rightarrow 0 . \tag{L}
\end{equation*}
$$

Following [9] we have
1.1. Theorem. (a) Sequence ( L ) is exact.
(b) There is a map $f: \mathbf{K}_{2 X} \rightarrow \mathbf{G}_{2 X}$ which is an isomorphism over $X \cdots\left\{x_{0}\right\}$,
(c) F is an acyclic sheaf.

Proof. (a) Surjectivity of $\partial$ follows from the surjectivity of $\partial$ in (GR*). At $x_{0}$ we have Image $T=\operatorname{Ker} \partial$ by definition of $\mathbf{F}$; at $x \neq x_{0}$ we use the exactness of (GR) to see that if $\partial(z)=0$, then $z \in$ Image $T$ locally at $x$.
(b) We let $j: K_{2}\left(X_{x_{0}}\right) \rightarrow\left(i_{X}\right)_{*} K_{2} k(X)$ and

$$
i: \coprod_{x \in X_{i}}\left(i_{x}\right)_{*} k(x)^{*} \rightarrow \coprod_{x \in X_{1}}\left(i_{x}\right)_{*} k(x)^{*}
$$

be the natural map;. Since $i R=T j$, there is a map of complexes $\left(\mathrm{GR}^{*}\right) \rightarrow(\mathrm{L})$ which
induces the morphism $f: \mathbf{K}_{2 X} \rightarrow \mathbf{G}_{2 X} ; f$ is an isomorphism over $X-\left\{x_{0}\right\}$ because GR is exact on the smooth locus.
(c) $\left(\mathrm{GR}^{*}\right)$ is an acyclic resolution of $\mathbf{K}_{2 X},[5]$. On the other hand $H^{2}\left(X, \mathbf{K}_{2 X}\right) \rightarrow$ $H^{2}\left(X, \mathrm{G}_{2 X}\right)$ by (b), hence $H^{0}\left(X, \amalg_{x \in X_{i}^{\prime}}\left(i_{x}\right)_{*} \mathbb{Z}_{x}\right) \rightarrow H^{2}\left(X, \mathrm{G}_{2 X}\right)$ is surjective. The acyclicity of $\mathbf{F}$ follows from this remark by looking at the spectral sequence of hypercohomology associated with (L).
1.2. Corollary. $H^{2}\left(X, \mathrm{~K}_{2 X}\right) \leadsto H^{2}\left(X, \mathrm{G}_{2 X}\right)$.

In order to prove Theorem 0.3 we shall prove
1.3. Theorem. $1^{0} H^{2}\left(X, \mathrm{G}_{2 X}\right)$ is finite.
1.4. Following Bloch [2, (5.4)] we consider the diagram:

where, according to our convention we set $m=l^{2}$. All columns and row in the diagram are exact but possibly for the first and last row.
1.5. Lemma. ${ }_{m} \mathbf{G}_{2}$ has cohomological dimension $\leq 1$.

Pror. Let $\mathbf{T}$ be the sheaf defined by exactness of the sequence

$$
0 \rightarrow{ }_{m} \mathbf{G}_{2} \rightarrow i_{X *}\left({ }_{m} K_{2} k(X)\right) \rightarrow \mathbf{T} \rightarrow 0,
$$

it suffices to prove that $T$ is acyclic. Note first the inclusion $T \hookrightarrow_{m} F$, because the first row in the diagram is exact on the left. Since $F \hookrightarrow \amalg_{x \in X_{1}} i_{x *}\left(k(x)^{*}\right)$, there is an inclusion

$$
i: \mathbf{T} \rightarrow \coprod_{x \in X_{1}} i_{x^{*}}\left(\mu_{m}\right)
$$

Now $i$ is an isomorphism over $X-\left\{x_{0}\right\}$, because $\mathbf{G}_{2 X}=\mathbf{K}_{2 X}$ there, hence in the following exact sequence the cokernel $\mathbf{C}$ is skyscraper, supported at $x_{0}$ :

$$
0 \rightarrow \mathbf{T} \rightarrow \coprod_{x \in X_{1}} i_{x+}\left(\mu_{m}\right) \xrightarrow{g} \mathbf{C} \rightarrow \mathbf{0} .
$$

To prove the acyclicity of $\mathbf{T}$ it suffices to show that $g$ is surjective on global sections. Such surjectivity can be verified in a standard way noting that the domain of $g$ is a flasque sheaf, while $H^{0}(X, \mathrm{C})=\mathrm{C}_{x_{0}}$, the stalk at $x_{0}$.
1.6. Lemma. There is an exact sequence

$$
0 \rightarrow H^{1}\left(X, \mathbf{G}_{2}\right) / m H^{1}\left(X, \mathbf{G}_{2}\right) \rightarrow H^{1}\left(X, \mathbf{G}_{2} / m \mathbf{G}_{2}\right) \rightarrow_{m} H^{2}\left(X, \mathbf{G}_{2}\right) \rightarrow 0 .
$$

Proof. Use the sequence of cohomology associated with the first column in the diagram.
1.7. In order to prove that ${ }_{m} H^{2}\left(X, \mathbf{G}_{2 X}\right)$ is finite we shall show that $H^{1}\left(X, \mathrm{G}_{2 X} / m \mathrm{G}_{2 X}\right)$ is finite.

Looking at the last row in the diagram, we write $\mathbf{S}^{(\nu)}$, or birefly $\mathbf{S}$, the sheaf image of $\mathbf{G}_{2} / m(v) \mathbf{G}_{2}$ in $i_{X *}\left(K_{2} k(X) / m(v) K_{2} k(X)\right)$. We set $p: \mathbf{G}_{2} / m(v) \mathbf{G}_{2} \rightarrow \mathbf{S}^{(\nu)}$ and $j: \mathrm{S}^{(v)} \rightarrow i_{X}\left(\mathrm{~K}_{2} k(X) / m(v) K_{2} k(X)\right)$, the obvious maps.
1.8. Lemma. $H^{1}\left(X, \mathrm{G}_{2 X} / m(v) \mathrm{G}_{2 X}\right) \widetilde{\rightarrow} H^{1}(X, \mathrm{~S})$.

Proof. The kernel of $p$ is skyscraper, supported at $x_{0}$, because $\mathbf{K}_{2}=\mathbf{G}_{2}$ over $X-\left\{x_{0}\right\}$ and $K_{2} / m \rightarrow i_{X *}\left(K_{2} k(X) / m\right)$ is injective there, [2, (5.4)].

We need later
1.9. Proposition. $0 \rightarrow \mathbf{S} \rightarrow i_{X *}\left(K_{2} k(X) / m\right) \rightarrow \mathbf{F} / m$ is an exact sequence.

Proof. Fix any point $y \in X$, let $a \in\left(i_{X *}\left(K_{2} k(X) / m\right)_{y}\right.$ with $\mathrm{d}(a)=0$. By exactness of the second column in the diagram $a=q_{0}(b), b \in\left(i_{X_{*}} K_{2} k(X)\right)_{y}$, while the hypothesis $\mathrm{d}(a)=0$ means $T(\dot{b})=m c, c \in(F)_{y}$. By exactness of the third row $0=\partial T(b)=$
$\partial(m c)=m \partial(c)$, hence $\partial(c)=0$ so that $c=T(g), g \in\left(i_{X_{*}}\left(K_{2} k(X)\right)\right)_{y}$. Therefore $T(m g)=T(b)$, hence $(b-m g) \in\left(\mathbf{G}_{2}\right)_{y}$ and $j p q(b-m g)=q_{0}(b)$, i.e. $a \in \mathbf{S}_{y}$.

Following Bloch's program for the non-singular case [3] we use no'v the sheaves $H^{q}\left(\mu_{m}^{\otimes 2}\right)$. They are the sheaves on $X$, in the Zariski topology, which are associated to the presheaf $U \rightarrow H_{\mathrm{et}}^{q}\left(U, \mu_{m}^{\otimes 2}\right)$, [4]. Let $R$ be the local ring $\mathbf{O}_{X, x_{0}}$, let $i: \mathrm{Sp} R \rightarrow X$ be the natural map, let $\mathrm{A}^{q}=i_{*}\left(i^{-1} \mathbf{H}^{q}\left(\mu_{m}^{\otimes 2}\right)\right.$.
1.10. Proposition. There is an exact sequence

$$
\begin{equation*}
0 \rightarrow \mathbf{H}^{q}\left(\mu_{m}^{\otimes 2}\right) \rightarrow \mathbf{A}^{q} \rightarrow \coprod_{x \in X_{i}^{*}} i_{x *} H^{q-1}\left(k(x), \mu_{m}^{\otimes 1}\right) \rightarrow \coprod_{x \in X_{:}^{*}} i_{x *} H^{q-2}\left(k(x), \mu_{m}^{\otimes 00}\right) \rightarrow \ldots \tag{+}
\end{equation*}
$$

and $\mathrm{A}^{q}$ is acyclic.
Proof. The proof given for (1.10) in [6] applies also here.
By definition of $\mathbf{H}^{q}$ the Leray spectral sequence associated with the morphism of topoi $\pi: X_{\text {et }} \rightarrow X_{\text {Zar }}$ is

$$
E_{2}^{p q}=H_{\mathrm{Zar}}^{q}\left(X, \mathbf{H}^{q}\left(\mu_{m}^{\otimes 2}\right) \Rightarrow H_{\mathrm{et}}^{p+q}\left(X, \mu_{m}^{\otimes 2}\right)\right.
$$

Now $H^{p}\left(X, \mathbf{H}^{q}\right)=0, p>q$, because $(+)$ is an acyclic resolution, hence $E_{2}^{12} \rightarrow$ $H_{\mathrm{et}}^{3}\left(X, \mu_{m}^{\otimes 2}\right)$ is an inclusion, therefore $E_{2}^{12}$ is finite, being contained in a finite group. To conclude the proof of 1.7 we show

### 1.11. Proposition. There is a surjective map

$$
H^{1}\left(X, \mathbf{H}^{2}\left(\mu_{m(v)}^{\otimes 2}\right)\right) \rightarrow H^{1}(X, \mathbf{S}) .
$$

Proof. We have a diagram


The map $\gamma$ comes fromi the inclusion $\|_{\text {a } \in X_{i}} K(x)^{*} \hookrightarrow \mathbf{F}$; the map $\beta$ is just the restriction map $H_{\mathrm{et}}^{2}(\mathrm{Sp} R) \rightarrow H_{\mathrm{et}}^{2}(\mathrm{Sp} k(X))$, recalling the basic Merkuriev-Suslin isomorphism $K_{2} k(X) / m \leadsto H^{2}\left(k(X), \mu_{m}^{\otimes 2}\right)$. The map $\alpha$ is defined using the exactness of the bottom row (Proposition 1.9). By the results of [3] and [7] for the nonsingular case we know that $\alpha$ is an isomorphism on $X-\left\{x_{0}\right\}$, hence the cokernel of $\alpha$ is acyclic.
2.

Using the arguments above we recover here a proposition of Levine [8], which extends a theorem of Roitman.
2.1. Proposition. If $Y$ is a complete surface with isolated singularities there is an isomorphism $\mathrm{CH}^{2}(Y)(l) \rightarrow \mathrm{Alb}\left(Y^{\prime}\right)(l)$, where $Y^{\prime}$ is the desingularization of $Y$ and $\mathrm{CH}^{2}(Y)(l)$ is the subgroup of elements of order a power of $l$.

Proof. As before let $X$ be the variety obtained from $Y$ by identifying all the singular points, let also $Y^{\prime \prime}$ be the normalization of $Y$, hence of $X$. We have to show $\mathrm{CH}^{\prime}(X)(l) \Longrightarrow \mathrm{Alb}\left(Y^{\prime}\right)(l)$.

Using the proper base change theorem and the Leray spectral sequence one has

$$
H_{\mathrm{et}}^{3}\left(X, \mathbb{Z} / l^{v}\right) \leftrightharpoons H_{\mathrm{el}}^{3}\left(Y, \mathbb{Z} / l^{v}\right) \leftrightharpoons H_{\mathrm{el}}^{3}\left(Y^{\prime \prime}, \mathbb{Z} / l^{v}\right),
$$

hence the same isomorphisms hold with $\mathbb{Z}_{i}$ coefficients. Further in the exact sequence

$$
0 \rightarrow T \rightarrow H_{\mathrm{el}}^{3}\left(Y^{\prime \prime}, \mathbb{Z}_{l}\right) \rightarrow H_{\mathrm{el}}^{3}\left(Y^{\prime}, \mathbb{Z}_{l}\right) \rightarrow 0
$$

the kernel $T$ is torsion, because the intersection matrix of the components of the exceptional divisor over a normal singularity is negative definite. Recall (see e.g. [7, $1.2(13)$ ]) the exact sequence

$$
0 \rightarrow H_{\mathrm{et}}^{3}\left(W, \mathbb{Z}_{l}\right) \otimes \mathbb{Q}_{l} / \mathbb{Z}_{l} \rightarrow H_{\mathrm{et}}^{3}\left(W, \mathbb{Q}_{l} / \mathbb{Z}_{l}\right) \rightarrow H_{\mathrm{el}}^{4}\left(W, \mathbb{Z}_{l}\right)_{\text {tors }} .
$$

If $W$ is an irreducible surface, then $H_{\mathrm{el}}^{4}\left(W, \mathbb{Z}_{1}\right)_{\text {ors }}=0$; therefore $H_{\mathrm{el}}^{3}\left(Y^{\prime \prime}, \mathbb{Z}_{l}\right) \otimes \mathbb{Q}_{l} / \mathbb{Z}_{l}=$ $H_{\mathrm{el}}^{3}\left(Y^{\prime \prime}, \mathbb{Q}_{l} / \mathbb{Z}_{i}\right)$. It follows that $H_{\mathrm{el}}^{3}\left(Y^{\prime}, \mathbb{Q}_{i} / \mathbb{Z}_{i}(2)\right)=H_{\mathrm{el}}^{3}\left(X, \mathbb{Q}_{i} / \mathbb{Z}_{i}(2)\right)$, where (2) indicates the Tate twist.
There is a diagram


Here $i$ is an isomorphism because $\mathbf{H}^{3}=0$ on a surface, since the cohomology groups on an affine variety vanish in dimension greater than the dimension of the variety. Taking direct limit one has


Here $a$ is the Albanese map; $h$ is defined so that the diagram is commutative; $f=a b$, [1]. Therefore $h$ is an isomorphism because $f$ is an isomo. phism, [2, (5.5)].

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